Integration Bee Mark Scheme

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Although these are not a complete set of methods, use these as a guide for your marking. Correct answers award full marks.

1. **A5**

$$\begin{aligned}\int (x - x\ln(x))^{-1}\,\mathrm{d}x &= -\int \frac{x^{-1}}{\ln(x)-1}\,\mathrm{d}x \\ &= -\ln(\ln(x)-1) + C\end{aligned}$$

2. **A5**

$$\begin{aligned}\int \operatorname{sech}(x)\,\mathrm{d}x &= \int \operatorname{sech}(x)\,\mathrm{d}x \\ &= \int \frac{\cosh(x)}{1-\sinh^2(x)}\,\mathrm{d}x \\ &= \arctan(\sinh(x)) + C\end{aligned}$$

3. **M2** (reduction to $\int \sec^3$) **M2** (reduction to $\int \sec$) **A1**

$$\begin{aligned}I = \int \sec^5(x)\,\mathrm{d}x &= \int \sec^3(x)(\tan(x))'\,\mathrm{d}x \\ &= \sec^3(x)\tan(x) - \int 3\sec^3(x)\tan^2(x)\,\mathrm{d}x \\ &= \sec^3(x)\tan(x) - 3\int \sec^5(x) - \sec^3(x)\,\mathrm{d}x \\ &= \sec^3(x)\tan(x) - 3I + 3\int \sec^3(x)\,\mathrm{d}x \\ I &= \frac{1}{4}\left(\sec^3(x)\tan(x) + 3\int \sec^3(x)\,\mathrm{d}x\right)\end{aligned}$$

By the application of similar algebra (i.e for a power of 3):

$$\begin{aligned}J = \int \sec^3(x)\,\mathrm{d}x &= \sec(x)\tan(x) - J + \int \sec(x)\,\mathrm{d}x \\ &= \frac{1}{2}\left(\sec(x)\tan(x) + \int \sec(x)\,\mathrm{d}x\right)\end{aligned}$$

The standard result is $\int \sec(x)\,\mathrm{d}x = \ln|\sec(x)+\tan(x)| + C$ but by the method in question 2 we have the neater $\int \sec(x)\,\mathrm{d}x = \operatorname{arsinh}(\tan(x)) + C$. We therefore have:

$$\begin{aligned}I &= \frac{1}{4}\left(\sec^3(x)\tan(x) + 3J\right) \\ &= \frac{1}{4}\left(\sec^3(x)\tan(x) + \frac{3}{2}\left(\sec(x)\tan(x) + \int \sec(x)\,\mathrm{d}x\right)\right) \\ &= \frac{1}{4}\left(\sec^3(x)\tan(x) + \frac{3}{2}\left(\sec(x)\tan(x) + \operatorname{arsinh}(\tan(x)) + C'\right)\right) \\ &= \frac{1}{4}\sec^3(x)\tan(x) + \frac{3}{8}\sec(x)\tan(x) + \frac{3}{8}\operatorname{arsinh}(\tan(x)) + C\end{aligned}$$

4. **M2** (Correct sub) **M2** (IBPs or reverse product rule) **A1**

$$\begin{aligned}
\int (1+\ln(x))\ln(\ln(x))\,\mathrm{d}x &\overset{\ln(x)\ \mapsto\ u}{=} \int e^u \ln(u) + e^u u \ln(u)\,\mathrm{d}u \\
&= \int e^u (u)' \ln(u) + (e^u)' u \ln(u) + e^u u (\ln(u))'\,\mathrm{d}x - \int e^u\,\mathrm{d}u \\
&= e^u u \ln(u) - e^u + C' \\
&= x\ln(x)\ln(\ln(x)) - x + C
\end{aligned}$$

5. **M1** (IBP) **M2** (constant added for IBPs)**M1** (Reverse chain rule) **A1**

$$\begin{aligned}
\int x^{n-1}\arctan\big(\sqrt{x^n-2}\big)\,\mathrm{d}x &= \int \left(\frac{x^n-1}{n}\right)' \arctan\big(\sqrt{x^n-2}\big)\,\mathrm{d}x \\
&= \left(\frac{x^n-1}{n}\right)\arctan\big(\sqrt{x^n-2}\big) - \int \left(\frac{x^n-1}{n}\right)\frac{\frac{n}{2}x^{n-1}}{\sqrt{x^n-2}(1+(x^n-2))}\,\mathrm{d}x \\
&= \left(\frac{x^n-1}{n}\right)\arctan\big(\sqrt{x^n-2}\big) - \int \left(\frac{x^n-1}{n}\right)\frac{\frac{n}{2}x^{n-1}}{\sqrt{x^n-2}(x^n-1)}\,\mathrm{d}x \\
&= \left(\frac{x^n-1}{n}\right)\arctan\big(\sqrt{x^n-2}\big) - \int \frac{x^{n-1}}{2\sqrt{x^n-2}}\,\mathrm{d}x \\
&= \left(\frac{x^n-1}{n}\right)\arctan\big(\sqrt{x^n-2}\big) - \frac{1}{n}\sqrt{x^n-2} + C
\end{aligned}$$

6. **M1** (Splitting the log) **M2** (complex solutions) **M1** (integral of log correct) **A1**

$$\begin{aligned}
\int \ln\left(x+\frac{1}{x}\right)\mathrm{d}x &= \int \ln\big(x^2+1\big) - \ln(x)\,\mathrm{d}x \\
&= -\frac{1}{x}\int \ln(x+i) + \ln(x-i)\,\mathrm{d}x
\end{aligned}$$

Using $\int \ln(x)\,\mathrm{d}x = x\ln(x) - x + C$:

$$\begin{aligned}
\int \ln\left(x+\frac{1}{x}\right)\mathrm{d}x &= \int \ln\big(x^2+1\big) - \ln(x)\,\mathrm{d}x \\
&= \int \ln(x+i) + \ln(x-i) - \ln(x)\,\mathrm{d}x \\
&= -x\ln(x) + x + (x-i)\ln(x-i) + (x+i)\ln(x+i) - 2x + C \\
&= -x\ln(x) - x + x\ln\big(x^2+1\big) + i\ln\left(\frac{x+i}{x-i}\right) + C \\
&= -x\ln(x) - x + x\ln\big(x^2+1\big) + 2i\,\mathrm{artanh}\left(\frac{i}{x}\right) + C \\
&= -x\ln(x) - x + x\ln\big(x^2+1\big) - 2\arctan\left(\frac{1}{x}\right) + C \\
&= -x\ln(x) - x + x\ln\big(x^2+1\big) - 2\arctan\left(\frac{1}{x}\right) + C
\end{aligned}$$

7. **M2** (Simplifies the integral) **M2** (Reverse chain rule) **A1**

$$\int \frac{1}{x\ln(xy)\ln\left(\frac{x}{y}\right)}\,\mathrm{d}x = \int \frac{x^{-1}}{\ln(x)^2 - \ln(y)^2}\,\mathrm{d}x$$
$$= -\frac{1}{\ln(y)}\operatorname{artanh}\left(\frac{\ln(x)}{\ln(y)}\right) + C$$

8. **M2** (suitable u sub to odd function) **A3**

$$\int_0^2 \ln^3\left(\frac{2}{x}-1\right)\mathrm{d}x \overset{e^u \mapsto \frac{2}{x}-1}{=} \int_\infty^{-\infty} -\frac{u^3}{2(e^u+1)^2}\,\mathrm{d}u$$
$$= 0$$

9. **M1** (reduction to sum) **M2** (simplification) **A2**

$$\int_0^\infty \frac{\lfloor x\rfloor^2}{\lfloor x\rfloor!}\cos(\pi\lfloor x\rfloor)\,\mathrm{d}x = \sum_{n=0}^{\infty}\frac{n^2}{n!}(-1)^n$$
$$= -\sum_{n=0}^{\infty}\frac{n^2}{n!}(-1)^n$$
$$= -\sum_{n=0}^{\infty}\left(\frac{n^2-n}{n!}+\frac{n}{n!}\right)(-1)^n$$
$$= -\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-2)!} - \sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!}$$
$$= e^{-1} - e^{-1}$$
$$= 0$$

10. **M2** (suitable u sub) **A3**

$$\int_0^\pi \cos(x+\tan(x))\,\mathrm{d}x \overset{u \mapsto \pi - x}{=} \int_0^\pi -\cos(x+\tan(x))\,\mathrm{d}x$$
$$= 0$$

11. **M1** (correct simplification) **M2** (Double angle formula) **M1** (IBPS) **A1**

$$\begin{aligned}
\int \arctan\left(\frac{1}{\sqrt{2}}\left(\frac{1}{x}-\frac{x}{2}\right)\right)\mathrm{d}x &= \frac{x\pi}{2} - \int \operatorname{arccot}\left(\frac{1}{\sqrt{2}}\left(\frac{1}{x}-\frac{x}{2}\right)\right)\mathrm{d}x \\
&= \frac{x\pi}{2} - \int \operatorname{arccot}\left(\frac{2-x^2}{2\sqrt{2}x}\right)\mathrm{d}x \\
&= \frac{x\pi}{2} - \int \arctan\left(\frac{1-\frac{x^2}{2}}{\sqrt{2}x}\right)\mathrm{d}x \\
&= \frac{x\pi}{2} - \int \arctan\left(\frac{\sqrt{2}x}{1-\frac{x^2}{2}}\right)\mathrm{d}x \\
&= \frac{x\pi}{2} - \int \arctan\left(\frac{2\frac{x}{\sqrt{2}}}{1-\left(\frac{x}{\sqrt{2}}\right)^2}\right)\mathrm{d}x
\end{aligned}$$

Note the following algebra:

$$\begin{aligned}
\tan(2t) &= \frac{2\tan(t)}{1-\tan(t)^2} \\
2t &= \arctan\left(\frac{2\tan(t)}{1-\tan(t)^2}\right)
\end{aligned}$$

The integrand is just $t = \arctan\left(\frac{x}{\sqrt{2}}\right)$:

$$\begin{aligned}
\frac{x\pi}{2} - \int \arctan\left(\frac{2\frac{x}{\sqrt{2}}}{1-\left(\frac{x}{\sqrt{2}}\right)^2}\right)\mathrm{d}x &= \frac{x\pi}{2} - \int 2\arctan\left(\frac{x}{\sqrt{2}}\right)\mathrm{d}x \\
&= \frac{x\pi}{2} - 2\int (x)'\arctan\left(\frac{x}{\sqrt{2}}\right)\mathrm{d}x \\
&= \frac{x\pi}{2} - 2x\arctan\left(\frac{x}{\sqrt{2}}\right) + 2\int \frac{\sqrt{2}x}{2+x^2}\,\mathrm{d}x \\
&= \frac{x\pi}{2} - 2x\arctan\left(\frac{x}{\sqrt{2}}\right) + \sqrt{2}\ln\left(x^2+2\right) + C
\end{aligned}$$

12. **M1** (IBPs) **M2** (factoring x^4+1) **A2**

$$
\begin{aligned}
I &= \int \ln\left(1+x^4\right) \mathrm{d}x \\
&= \int (x') \ln\left(x^4+1\right) \mathrm{d}x \\
&= x\ln\left(x^4+1\right) - \int \frac{4x^4}{x^4+1} \mathrm{d}x \\
&= x\ln\left(x^4+1\right) - 4x + \int \frac{4}{x^4+1} \mathrm{d}x \\
&= x\ln\left(x^4+1\right) - 4x + \int \frac{\sqrt{2}x+2}{x^2+\sqrt{2}x+1} - \frac{\sqrt{2}x-2}{x^2-\sqrt{2}x+1} \mathrm{d}x \\
&= x\ln\left(x^4+1\right) - 4x + \frac{1}{\sqrt{2}}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \int \frac{1}{x^2+\sqrt{2}x+1} + \frac{1}{x^2-\sqrt{2}x+1} \mathrm{d}x \\
&= x\ln\left(x^4+1\right) - 4x + \frac{1}{\sqrt{2}}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \int \frac{1}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} + \frac{1}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} \mathrm{d}x \\
&= x\ln\left(x^4+1\right) - 4x + \frac{1}{\sqrt{2}}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \sqrt{2}\left(\arctan\left(x\sqrt{2}-1\right) + \arctan\left(x\sqrt{2}+1\right)\right) + C
\end{aligned}
$$

13. **M2** (suitable u sub) **M1** (simplification) **A2**

$$
\begin{aligned}
\int \sec(x)\operatorname{arsinh}(\tan(x))\,\mathrm{d}x &\overset{x\ \mapsto\ \arctan(\sinh(u))}{=} \int u\operatorname{sech}(u)\,\mathrm{d}u \int u\operatorname{sech}(u)\sec(\arctan(\sinh(u)))\,\mathrm{d}u \\
&= \int u\operatorname{sech}(u)\sqrt{1+\tan^2(\arctan(\sinh(u)))}\,\mathrm{d}u \\
&= \int u\operatorname{sech}(u)\sqrt{1+\sinh^2(u)}\,\mathrm{d}u \\
&= \int u\operatorname{sech}(u)\cosh(u)\,\mathrm{d}u \\
&= \int u\,\mathrm{d}u \\
&= \frac{u^2}{2} + C \\
&= \frac{(\operatorname{arsinh}(\tan(x)))^2}{2} + C
\end{aligned}
$$

14. **M1** (Rewriting artanh) **M2** (Simplification) **A2**

$$\begin{aligned}
\int \frac{e^{-2\operatorname{artanh}(\tan(x))}}{\ln(1+\sin(2x))}\,\mathrm{d}x &= \int \frac{e^{-\ln\left(\frac{1+\tan(x)}{1-\tan(x)}\right)}}{\ln(1+\sin(2x))}\,\mathrm{d}x \\
&= \int \frac{\left(\frac{1-\tan(x)}{1+\tan(x)}\right)}{2\ln(\cos(x)+\sin(x))}\,\mathrm{d}x \\
&= \int \frac{\left(\frac{\cos(x)-\sin(x)}{\cos(x)+\sin(x)}\right)}{2\ln(\cos(x)+\sin(x))}\,\mathrm{d}x \\
&= \frac{1}{2}\ln(\ln(\cos(x)+\sin(x)))+C
\end{aligned}$$

15. **M1** (reduction to a sum) **M3** (Finding $\zeta(2)$) **A1**

$$\begin{aligned}
\int_2^\infty \frac{2}{\lfloor x\rfloor^2(\lfloor x\rfloor+2)} - \cosh^2(\operatorname{artanh}\lfloor x\rfloor)\,\mathrm{d}x &= \sum_{n=2}^{\infty}\frac{2}{n^2(n+2)} - \cosh^2(\operatorname{artanh}(n)) \\
&= \sum_{n=2}^{\infty}\frac{2}{n^2(n+2)} - \frac{1}{\operatorname{sech}^2(\operatorname{artanh}(n))} \\
&= \sum_{n=2}^{\infty}\frac{2}{n^2(n+2)} - \frac{1}{1-\tanh^2(\operatorname{artanh}(n))} \\
&= \sum_{n=2}^{\infty}\frac{2}{n^2(n+2)} - \frac{1}{1-n^2} \\
&= \frac{1}{3} + \sum_{n=3}^{\infty}\frac{2}{(n-1)^2(n+1)} - \frac{1}{1-n^2} \\
&= \frac{1}{3} + \sum_{n=3}^{\infty}\frac{n+1}{(n-1)^2(n+1)} \\
&= \frac{1}{3} + \sum_{n=2}^{\infty}\frac{1}{n^2} \\
&= \frac{1}{3} + \zeta(2) - 1 \\
&= \zeta(2) - \frac{2}{3} \\
&= \frac{\pi^2}{6} - \frac{2}{3}
\end{aligned}$$

16. **M1** (Expanding sine) **M1** (Spotting Gamma) **M1** (solving factorials) **M1** (Spotting arctan) **A1**

$$\begin{aligned}
\int_0^\infty \frac{e^{-x}}{x}\sin\left(\frac{x}{2}\right)\mathrm{d}x &= \int_0^\infty \frac{e^{-x}}{x}\sum_{n=0}^\infty \frac{(-1)^n}{2n+1}\left(\frac{x}{2}\right)^{2n+1}\mathrm{d}x \\
&= \int_0^\infty \sum_{n=0}^\infty \frac{2^{-(2n+1)}(-1)^n}{(2n+1)!}e^{-x}x^{2n}\,\mathrm{d}x \\
&= \sum_{n=0}^\infty \frac{2^{-(2n+1)}(-1)^n}{(2n+1)!}\int_0^\infty e^{-x}x^{2n}\,\mathrm{d}x \\
&= \sum_{n=0}^\infty \frac{2^{-(2n+1)}(-1)^n}{(2n+1)!}\Gamma(2n+1) \\
&= \sum_{n=0}^\infty \frac{2^{-(2n+1)}(-1)^n}{(2n+1)!}(2n)! \\
&= \sum_{n=0}^\infty \frac{2^{-(2n+1)}(-1)^n}{2n+1} \\
&= \arctan\left(\frac{1}{2}\right)
\end{aligned}$$

17. **M2** (Suitable u-sub) **M2** (Expanding arctan) **A1**

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \tan(x)\arctan(\cos(x))\,\mathrm{d}x &\overset{u\ \mapsto\ \sec(x)}{=} \int_1^\infty \frac{1}{u}\arctan\left(\frac{1}{u}\right)\mathrm{d}u \\
&= \int_1^\infty \frac{1}{u}\sum_{n=0}^\infty \frac{u^{-(2n+1)}(-1)^n}{2n+1}\,\mathrm{d}u \\
&= \int_1^\infty \sum_{n=0}^\infty \frac{u^{-(2n+2)}(-1)^n}{2n+1}\,\mathrm{d}u \\
&= \sum_{n=0}^\infty \int_1^\infty \frac{u^{-(2n+2)}(-1)^n}{2n+1}\,\mathrm{d}u \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^2}\,\mathrm{d}u \\
&= G
\end{aligned}$$

18. **M2** (Weierstrass sub) **M1** (Correct parity $[(-1)^{n+k}]$) **M1** (Sub to Beta function) **A1**

$$\begin{aligned}
\int_0^1 \frac{(2x)^k(1+x^2)^{n-1}}{(x^2-1)^{n+k}}\,\mathrm{d}x &= \frac{1}{2}\int_0^1 \frac{2}{1+x^2}\left(\frac{1+x^2}{x^2-1}\right)^n\left(\frac{2x}{x^2-1}\right)^k \mathrm{d}x \\
&\overset{x=\tan\left(\frac{t}{2}\right)}{=} \frac{1}{2}\int_0^1 (-\tan t)^k(-\sec(t))^n\,\mathrm{d}t \\
&= \frac{(-1)^{n+k}}{2}\int_0^{\frac{\pi}{2}} (\tan^2(t))^{\frac{k}{2}}(\sec^2(t))^{\frac{n}{2}}\,\mathrm{d}t \\
&= \frac{(-1)^{n+k}}{2}\int_0^{\frac{\pi}{2}} (\sin^2(t))^{\frac{k}{2}}(\cos^2(t))^{-\frac{n+k}{2}}\,\mathrm{d}t \\
&= \frac{(-1)^{n+k}}{2}\int_0^{\frac{\pi}{2}} (\sin^2(t))^{\frac{k}{2}}(1-\sin^2(t))^{-\frac{n+k}{2}}\,\mathrm{d}t \\
&= \frac{(-1)^{n+k}}{2}\int_0^{\frac{\pi}{2}} (\sin^2(t))^{\frac{k}{2}}(1-\sin^2(t))^{-\frac{n+k}{2}}\,\mathrm{d}t \\
&\overset{u\ \mapsto\ \sin^2(t)}{=} (-1)^{n+k}\frac{1}{4}\int_0^1 \frac{u^{\frac{k}{2}}(1-u)^{-\frac{n+k}{2}}}{\sqrt{(1-u)u}}\,\mathrm{d}u \\
&= \frac{(-1)^{n+k}}{4}\int_0^1 u^{\frac{k-1}{2}}(1-u)^{-\frac{1+n+k}{2}}\,\mathrm{d}u \\
&= \frac{(-1)^{n+k}}{4}B\left(\frac{k+1}{2},\frac{1-n-k}{2}\right)
\end{aligned}$$

19. **M2** (Correct rewriting of integral) **M2** (Wallis Product) **A1**

$$\begin{aligned}
\int_1^\infty \frac{1}{x^2}\frac{2}{\pi}\arcsin\left|\sin\left(\frac{\pi}{2}x\right)\right|\mathrm{d}x &= \sum_{n=1}^\infty \int_{2n-1}^{2n}\frac{(2n-x)}{x^2}\,\mathrm{d}x + \int_{2n}^{2n+1}\frac{(x-2n)}{x^2}\,\mathrm{d}x \\
&= \sum_{n=1}^\infty \frac{1}{2n-1} + \ln\left(1-\frac{1}{2n}\right) - \frac{1}{2n+1} + \ln\left(1+\frac{1}{2n}\right) \\
&= \sum_{n=1}^\infty \frac{1}{2n-1} - \frac{1}{2n+1} + \ln\left(1-\frac{1}{4n^2}\right) \\
&= \sum_{n=1}^\infty \frac{1}{2n-1} - \frac{1}{2n+1} - \ln\left(\frac{4n^2}{4n^2-1}\right) \\
&= \sum_{n=1}^\infty \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) - \ln\left(\prod_{i=1}^\infty \frac{4i^2}{4i^2-1}\right) \\
&= -\ln\left(\frac{\pi}{2}\right) + \sum_{n=1}^\infty\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\
&= -\ln\left(\frac{\pi}{2}\right) + \frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7}\dots \\
&= 1 - \ln\left(\frac{\pi}{2}\right)
\end{aligned}$$

20. **M2** (Correct DUTIS) **M2** (Correct integration) **A1**

$$\begin{aligned}
\int_1^e \frac{\ln(x)}{x} \ln^n(\ln(x))\,\mathrm{d}x &= \int_1^e \frac{1}{x}\frac{\partial^n}{\partial a^n}\left[\ln(x)^a\right]\mathrm{d}x \\
&= \frac{\mathrm{d}^n}{\mathrm{d}a^n}\left[\int_1^e \frac{\ln(x)^a}{x}\,\mathrm{d}x\right] \\
&= \frac{\mathrm{d}^n}{\mathrm{d}a^n}\left[\left[\frac{\ln(x)^{a+1}}{a+1}\right]_1^e\right] \\
&= \frac{\mathrm{d}^n}{\mathrm{d}a^n}\left[\frac{1}{a+1}\right] \\
&= \frac{n!}{(a+1)^{n+1}}(-1)^n \\
&= \frac{n!}{2^{n+1}}(-1)^n
\end{aligned}$$